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Path integral for one-dimensional Dirac oscillator

R. Rekioua, T. Boudjedaa[a]

Laboratoire de Physique Théorique, Faculté des Sciences, Université de Jijel, BP 98, Ouled Aissa, Jijel 18000, Algeria

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Abstract. In this paper we derive the propagator for the one-dimensional Dirac oscillator using the supersymmetric path integral formalism. The spin calculations are carried out with the help of the technique of Grassmann functional integration. The Green function is exactly evaluated. The Polyakov spin factor is explicitly derived and the energy spectrum and the corresponding wave functions are deduced.

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1 Introduction

It is obvious that the path integral for the Dirac electron still encounters some problems. This is connected both to the discrete nature of the spin and to the invariance of the velocity of light c in inertial frames. For example, in the operator formalism, a direct calculation shows that the eigenvalues of the velocity operator are equal to $\pm c$ because this velocity operator is related to the spin matrices which have discrete eigenvalues. From the relativistic invariance principle, if the particle had this velocity $\pm c$, its mass would be null, which is not however the case of electron physics. Moreover, the Feynman path integral uses classical quantities like a continuous trajectory, and for these discrete eigenvalues it is not possible to associate continuous paths with them. In short, it is not obvious how one could propose a model for the Dirac electron satisfying the relativistic invariance requirements and describing the discrete nature of the spin. Actually, there have been several attempts combining such requirements; one of them is the supersymmetric model based on the Grassmann variables proposed firstly by Fradkin [1], then re-examined by Berezin and Marinov [2, 3], and recently taken up again by Fradkin and Gitman [4]. They succeeded to give a straightforward way of constructing the corresponding relativistic Klein–Gordon and Dirac propagators. The fundamental idea of this formalism is to write formally the causal Green function like the inverse of an operator and then, by means of an integral representation, expressing this inverse as a standard Schrödinger evolution operator. In the case of the Klein–Gordon equation, only one time of evolution is used, known as the Schwinger proper time, and in the case of the Dirac equation, one uses a generalized proper time. It is a supersymmetric proper time having two parts, one bosonic and the other fermionic. The bosonic part is exactly that of Schwinger and the fermionic one is responsible of the projection of the Klein–Gordon states on those of the Dirac ones. This formalism was developed from the calculative point of view in the case of some concrete applications [5–11]. These latter are particular configurations of external fields such as a constant field, a plane wave field and their combination. However, to our knowledge, nothing was done in the case of the potentials with spherical symmetry. The major difficulty lies in the property of composition by the kinetic moment and spin, known as the spin–orbit coupling.

Our aim in this paper is to adapt this formalism in the case of the Dirac oscillator. This physical system was first introduced by Moshinsky and Szczepaniak [12]. It attracted much attention because of its various physical applications. For example, the Moshinsky and Smirnov book devotes several chapters to these applications [13]. Rosmej and Arvieu [14] have shown a very interesting analogy between the relativistic Dirac oscillator and the Jaynes–Cummings model. This model has also been studied in connection with supersymmetric relativistic quantum mechanics, quark confinement models in quantum chromodynamics and other relativistic conformally invariant problems.

In this paper, we derive the path integral representation for one-dimensional Dirac oscillator. The construction is analogous to that of the Dirac particle with an anomalous magnetic moment [15]. In Sect. 2, we present the derivation of the Dirac oscillator propagator via the supersymmetric formalism proposed by Fradkin and Gitman. In Sect. 3, we determine exactly the Green function expression by doing explicitly all the integrations over the spin variables using the Grassmann functional integration technique, and therefore we obtain the explicit expression of the Polyakov spin factor. In Sect. 4, we carry out the remaining bosonic path integral. The energy spectrum and the corresponding eigenfunctions, for the positive- and negative-energy states, are obtained. Section 5 is devoted to the conclusion.

[a] e-mail: boudjeda@caramail.com

2 Derivation of path integral for the one-dimensional Dirac oscillator

In this section, we present a derivation of the path integral for the one-dimensional Dirac oscillator propagator following the Fradkin–Gitman method [4, 15]. The propagator of the one-dimensional Dirac oscillator is the causal Green function $S^{c}(x,y)$ of the following Dirac–Pauli equation:

$$(\hat{\not{\pi}} - mc)S^{c}(x,y) = -\delta(x-y)\,, \quad m \to m - \mathrm{i}\varepsilon\,, \tag{1}$$

where

$$\hat{\not{\pi}} = \gamma^{\mu}\hat{\pi}_{\mu}\,,$$

where $\hat{\pi}_{\mu}$ has components

$$\hat{\pi}_0 = \mathrm{i}\hbar\partial_0 = \frac{\mathrm{i}\hbar}{c}\frac{\partial}{\partial t}, \qquad \pi_i = \mathrm{i}\hbar\partial_i - \mathrm{i}m\omega\gamma^0 x_i = \mathrm{i}\hbar\frac{\partial}{\partial x^i} - \mathrm{i}m\omega\gamma^0 x_i\,. \tag{2}$$

Multiplying both sides of (1) by $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, we get

$$(\hat{\not{\pi}} - mc\gamma^5)\widetilde{S}^{c}(x,y) = \delta(x-y)\,, \tag{3}$$

with

$$\widetilde{S}^{c}(x,y) = S^{c}(x,y)\gamma^5\,, \quad \tilde{\gamma}^5 = \gamma^5\,, \quad \tilde{\gamma}^{\mu} = \gamma^5\gamma^{\mu}\,, \qquad \hat{\pi}_{\mu}\left(\hat{\pi}_0 = \mathrm{i}\hbar\partial_0\,, \quad \pi_i = \mathrm{i}\hbar\partial_i + \mathrm{i}m\omega\gamma^5\tilde{\gamma}^0 x_i\right). \tag{4}$$

The γ-matrices verify the standard commutation relations

$$[\tilde{\gamma}^n, \tilde{\gamma}^m]_+ = 2\eta^{nm}. \tag{5}$$

First, we present $\widetilde{S}^{c}(x,y)$ as a matrix element of an operator $\widetilde{S}^{c}$,

$$\widetilde{S}^{c}(x,y) = \langle x|\widetilde{S}^{c}|y\rangle\,, \tag{6}$$

where $\widetilde{S}^{c}$ is given by

$$\widetilde{S}^{c} = \frac{I}{\hat{\not{\pi}} - mc\gamma^5} \tag{7}$$

and write $\widetilde{S}^{c}$ like a pure Fermi operator

$$\widetilde{S}^{c} = (\hat{\tilde{\not{\pi}}} - mc\gamma^5)\frac{1}{(\hat{\not{\pi}} - mc\gamma^5)(\hat{\tilde{\not{\pi}}} - mc\gamma^5)}\,, \tag{8}$$

where

$$(\hat{\not{\pi}} - mc\gamma^5) = \hat{\not{p}} - \mathrm{i}m\omega\gamma^5\tilde{\gamma}^0\tilde{\gamma}\cdot\hat{\mathbf{x}} - mc\gamma^5, \qquad (\hat{\tilde{\not{\pi}}} - mc\gamma^5) = \hat{\not{p}} + \mathrm{i}m\omega\gamma^5\tilde{\gamma}^0\tilde{\gamma}\cdot\hat{\mathbf{x}} - mc\gamma^5. \tag{9}$$

So,

$$\widetilde{S}^{c} = (\hat{\tilde{\not{\pi}}} - mc\gamma^5)_{\mathrm{out}}\widetilde{G}^{c}(x,y)\,,$$

with

$$\widetilde{G}^{c}(x,y) = \left\langle x\left|\frac{1}{(\hat{\not{\pi}} - mc\gamma^5)(\hat{\tilde{\not{\pi}}} - mc\gamma^5)}\right|y\right\rangle. \tag{10}$$

According to the Schwinger proper-time method, $\widetilde{G}^{c}(x,y)$ is written as

$$\widetilde{G}^{c}(x,y) = \left(-\frac{\mathrm{i}}{\hbar}\right)\int \mathrm{d}\lambda\langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(\lambda)}|y\rangle\,, \tag{11}$$

where

$$\hat{H}(\lambda) = -\lambda(\hat{\not{\pi}} - mc\gamma^5)(\hat{\tilde{\not{\pi}}} - mc\gamma^5) = \lambda[-\hat{p}^2 + m^2c^2 + m^2\omega^2\hat{x}^2 - m\omega\hbar\gamma^5\tilde{\gamma}^0]\,. \tag{12}$$

In order to derive a path integral representation for $\widetilde{G}^{c}(x,y)$, we follow the standard discretization method for the kernel of (11).

As usually done, we write $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(\lambda)\right) = \exp\left(-\frac{\mathrm{i}}{\hbar}\times\frac{\hat{H}}{N+1}\right)^{N+1}$ and insert N times the identity $\int|x\rangle\langle x|\,\mathrm{d}x = 1$ between all operators $\exp\left(-\frac{\mathrm{i}}{\hbar}\frac{\hat{H}}{N+1}\right)$ and $(N+1)$ additional integrations over λ to transform the expression of $\widetilde{G}^{c}(x,y)$ into the following path integral:

$$\widetilde{G}^{c}(x,y) = \lim_{N\to+\infty}\left(-\frac{\mathrm{i}}{\hbar}\right)\int \mathrm{d}\lambda_0\int\prod_{k=1}^{N+1}\mathrm{d}\lambda_k\int\prod_{k=1}^{N+1}\mathrm{d}x_k \times\prod_{k=1}^{N+1}\left[\left\langle x_k\left|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(\lambda_k)\Delta\tau}\right|x_{k-1}\right\rangle\delta(\lambda_k-\lambda_{k-1})\right]\,, \tag{13}$$

where

$$\Delta\tau = \frac{1}{N+1}\,, \quad x_0 = x_{\mathrm{a}}\,, \quad x_{N+1} = x_{\mathrm{b}}\,. \tag{14}$$

Expanding, as usual, the matrix elements of (13) to the first order in $\Delta\tau$ and then inserting $(N+1)$ times the identity $\int|p\rangle\langle p|\,\mathrm{d}p = 1$, the Green function is written, using the mid-point prescription, as

$$\widetilde{S}^{c}(x,y) = \left(-\frac{\mathrm{i}}{\hbar}\right)(\hat{\tilde{\not{\pi}}} - mc\gamma^5)_{\mathrm{out}}\lim_{N\to+\infty}\mathcal{T} \times\int\mathrm{d}\lambda_0\int\prod_{k=1}^{N+1}\mathrm{d}\lambda_k\int\prod_{k=1}^{N+1}\mathrm{d}x_k\int\prod_{k=1}^{N+1}\frac{\mathrm{d}p_k}{(2\pi\hbar)^2} \times\prod_{k=1}^{N+1}\exp\left[\frac{\mathrm{i}}{\hbar}\left(p_k\frac{x_k-x_{k-1}}{\Delta\tau} - \hat{H}(\lambda_k,\bar{x}_k,p_k)\right)\Delta\tau\right] \times\delta(\lambda_k-\lambda_{k-1})\,, \tag{15}$$

or by using for the delta function $\delta(\lambda_k - \lambda_{k-1})$ the habitual integral representation

$$\widetilde{S}^{\mathrm{c}}(x,y) = \left(-\frac{\mathrm{i}}{\hbar}\right)(\widehat{\tilde{\not\pi}} - mc\gamma^5)_{\mathrm{out}} \times \mathcal{T}\int \mathrm{d}\lambda_0 \int \mathrm{D}x \int \mathrm{D}p \int \mathrm{D}\lambda \int \mathrm{D}\pi_\lambda \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^1 \left[\lambda(p^2 - m^2c^2 - m^2\omega^2x^2 + m\omega\hbar\tilde{\gamma}^5\tilde{\gamma}^0) + p\dot{x} + \pi_\lambda\dot{\lambda}\right]\mathrm{d}\tau\right\}, \tag{16}$$

where $x^\mu(\tau)$, $\lambda(\tau)$, $\pi_\lambda(\tau)$ are even trajectories, obeying the boundary conditions

$$x(0) = x_{\mathrm{in}}\,, \quad y(1) = x_{\mathrm{out}}\,, \quad \lambda(0) = \lambda_0\,.$$

The ordering operator $\mathcal{T}$ acts on the γ-matrices which are supposed formally to be depending on the time parameter τ.

By means of the source technique, (16) can be transformed as follows:

$$\widetilde{S}^{\mathrm{c}}(x,y) = \left(-\frac{\mathrm{i}}{\hbar}\right)(\widehat{\tilde{\not\pi}} - mc\gamma^5)_{\mathrm{out}}\mathcal{T} \times \int \mathrm{d}\lambda_0 \int \mathrm{D}x \int \mathrm{D}p \int \mathrm{D}\lambda \int \mathrm{D}\pi_\lambda \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^1 \left[\lambda\left(p^2 - m^2c^2 - m^2\omega^2x^2 + m\omega\hbar\frac{\delta_{\mathrm{l}}}{\delta\rho_5}\frac{\delta_{\mathrm{l}}}{\delta\rho_0}\right) + p\dot{x} + \pi_\lambda\dot{\lambda}\right]\mathrm{d}\tau\right\} \times \exp\left\{\int_0^1 \rho_n(\tau)\tilde{\gamma}^n\,\mathrm{d}\tau\right\}\Bigg|_{\rho=0}, \tag{17}$$

where $\rho_n(\tau)$ are odd sources, and by definition they anticommute with the γ-matrices.

We now present the quantity $\mathcal{T}\exp\{\int_0^1 \rho_n(\tau)\tilde{\gamma}^n\,\mathrm{d}\tau\}$ as a path integral over odd trajectories via the following functional formula:

$$\mathcal{T}\exp\left\{\int_0^1 \rho_n(\tau)\tilde{\gamma}^n\,\mathrm{d}\tau\right\}\Bigg|_{\rho=0} = \exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta_{\mathrm{l}}}{\delta\theta^n}\right) \times \int_{\Psi(0)+\Psi(1)=\theta} \exp\left[\int_0^1 \left(\Psi_n\dot{\Psi}^n - 2\mathrm{i}\rho_n\Psi^n\right)\mathrm{d}\tau + \Psi_n(1)\Psi^n(0)\right]\mathfrak{D}\Psi\Bigg|_{\theta=0}^{\rho=0}, \tag{18}$$

where

$$\mathfrak{D}\Psi = D\Psi\left[\int_{\Psi(0)+\Psi(1)=\theta} \mathrm{D}\Psi \exp\left(\int_0^1 \Psi_n\dot{\Psi}^n\,\mathrm{d}\tau\right)\right]^{-1}, \tag{19}$$

and θ^n and $\Psi^n(\tau)$ are Grassmann (odd) variables, anticommuting with the γ-matrices; $\frac{\delta_{\mathrm{l}}}{\delta\theta^n}$ stands for left derivative over the Grassmann variables. In addition, these odd trajectories $\Psi^n(\tau)$ obey the anti-periodic boundary conditions

$$\Psi^n(1) + \Psi^n(0) = \theta^n. \tag{20}$$

Rewriting the $\Psi^5\Psi^0$ term as follows:

$$\Psi^5\Psi^0 = \frac{1}{2}\Psi^n\Psi^m\mathcal{F}_{nm}\,, \quad n = 0,1,5, \tag{21}$$

and using (18) we get the following Hamiltonian path integral representation for $\widetilde{G}^{\mathrm{c}}(x,y)$:

$$\widetilde{G}^{\mathrm{c}}(x,y) = \left(-\frac{\mathrm{i}}{\hbar}\right)\exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta_{\mathrm{l}}}{\delta\theta^n}\right)\int \mathrm{d}\lambda_0 \times \int \mathrm{D}x \int \mathrm{D}p \int \mathrm{D}\pi_\lambda \int \mathrm{D}\lambda \int_{\Psi(0)+\Psi(1)=\theta} \mathfrak{D}\Psi \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^1 \left[\lambda\big(p^2 - m^2c^2 - m^2\omega^2x^2 - 2m\omega\hbar\Psi^n\Psi^m\mathcal{F}_{nm}\big) + p\dot{x} + \pi_\lambda\dot{\lambda} - \mathrm{i}\hbar\Psi_n\dot{\Psi}^n\right]\mathrm{d}\tau + \Psi_n(1)\Psi^n(0)\right\}_{\theta=0}, \tag{22}$$

where $\mathcal{F}_{nm}$ $(n = 0,1,5)$ is, being an antisymmetric matrix, defined by

$$\mathcal{F}_{50} = I\,. \tag{23}$$

To get the Lagrangian path integral representation, we make the shift $p^\mu \longrightarrow p^\mu - \frac{\dot{x}}{e}$, we then have

$$\widetilde{G}^{\mathrm{c}}(x,y) = \left(\frac{-\mathrm{i}}{\hbar}\right)\exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta_{\mathrm{l}}}{\delta\theta^n}\right) \times \int \mathrm{d}\lambda_0 \int \mathrm{D}\lambda \int \mathrm{D}\pi_\lambda \int \mathrm{D}x \int_{\Psi(0)+\Psi(1)=\theta} \mathfrak{D}\Psi \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^1 \left[\frac{-1}{4\lambda}(\dot{x})^2 - \lambda\big(m^2c^2 + m^2\omega^2x^2 + 2\Psi^n\Psi^m\mathcal{F}_{nm}\big) + \pi_e\dot{e} - \mathrm{i}\hbar\Psi_n\dot{\Psi}^n\right]\mathrm{d}\tau + \Psi_n(1)\Psi^n(0)\right\}_{\theta=0}. \tag{24}$$

In order to be free from the boundary conditions $\Psi^n(1) + \Psi^n(0) = \theta^n$, it is suitable to replace the integration over Ψ by that of the odd velocities ω, following the replacement

$$\Psi(\tau) = \frac{1}{2}\int_0^1 \varepsilon(\tau - \acute{\tau})\omega(\acute{\tau})\,\mathrm{d}\acute{\tau} + \frac{\theta}{2}\,, \quad \varepsilon(\tau) = \mathrm{sign}(\tau) \quad \dot{\Psi}^n(\tau) = \omega(\tau)\,, \quad \Psi_n(1)\Psi^n(0) = -\frac{\theta^n}{2}\int_0^1 \omega^n(\tau)\,\mathrm{d}\tau\,. \tag{25}$$

Obviously, the domain of integration over ω is now free, i.e., the boundary conditions are automatically satisfied.

Thus, we obtain

$$\begin{aligned}\widetilde{S}^{\mathrm{c}}(x,y)=&\left(\frac{-\mathrm{i}}{\hbar}\right)(\widehat{\tilde{\not{\pi}}}-mc\gamma^5)_{\mathrm{out}}\\&\times\exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta_{\mathrm{l}}}{\delta\theta^n}\right)\int\mathrm{d}\lambda\int\mathrm{D}x\int\mathfrak{D}\omega\\&\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^1\left[\frac{-1}{4\lambda}(\dot{x})^2-\lambda(m^2c^2+m^2\omega^2x^2)\right]\mathrm{d}\tau_1\right.\\&-\frac{1}{2}\int_0^1\int_0^1\omega^n(\tau_1)\varepsilon(\tau_1-\tau_2)\omega_n(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\\&+\frac{\alpha}{2}\int_0^1\int_0^1\int_0^1\omega^n(\tau_1)\varepsilon(\tau_1-\tau_2)\mathcal{F}_{nm}(\tau_2)\\&\quad\times\varepsilon(\tau_2-\tau_3)\omega^m(\tau_3)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3\\&-\alpha\theta^n\int_0^1\int_0^1\mathcal{F}_{nm}(\tau_1)\varepsilon(\tau_1-\tau_2)\omega^m(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\\&\left.-\frac{\alpha}{2}\theta^n\theta^m\int_0^1\mathcal{F}_{nm}(\tau_1)\,\mathrm{d}\tau_1\right\}_{\theta=0},\end{aligned}\qquad(26)$$

where $\alpha=\mathrm{i}m\omega\lambda$.

Actually, the integrations over (π_λ,λ) have been directly done giving $\lambda=\lambda_0$.

To end this section, let us show that this Green function $\widetilde{S}^{\mathrm{c}}(x,y)$ given by (26) verifies (3). First, it is preferable to introduce an analogy with Schrödinger propagation using the Fock trick, known as the Schwinger proper-time technique. Let us write initially for (10) the following decomposition [18]:

$$\begin{aligned}\widetilde{G}^{\mathrm{c}}(x,y)&=\widetilde{G}^{\mathrm{c}}(x,y)\\&=\left(-\frac{\mathrm{i}}{\hbar}\right)\int_0^{+\infty}\mathrm{d}\lambda\exp\left(\frac{\mathrm{i}\lambda m^2c^2}{\hbar}\right)\widetilde{K}(x,y,\lambda),\end{aligned}\qquad(27)$$

where $\frac{m^2c^2}{\hbar}$ plays the role of energy corresponding to the proper time λ. It is easy to check that

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial\lambda}-\widetilde{H}\right)\widetilde{K}(x,y,\lambda)=\delta(x-y)\delta(\lambda),\qquad(28)$$

with

$$\widetilde{H}=[-\hat{p}^2+m^2\omega^2\hat{x}^2-m\omega\hbar\gamma^5\tilde{\gamma}^0].\qquad(29)$$

The close resemblance of (28) to the Schrödinger equation means that we can now use $\widetilde{K}(x,y,\lambda)$ to describe the following propagation:

$$\Phi(x,\lambda)=\int\widetilde{K}(x,y,\lambda)\Phi(y,0)\,\mathrm{d}y,\qquad(30)$$

where

$$\begin{aligned}\widetilde{K}(x,y,\lambda)=&\exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta_{\mathrm{l}}}{\delta\theta^n}\right)\int\mathrm{D}x\int\mathrm{D}p\,\mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\int_0^1\lambda p^2\,\mathrm{d}\tau\right)}\\&\times\int\mathfrak{D}\omega\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^1\left[\frac{-1}{4\lambda}\dot{x}^2-\lambda m^2\omega^2x^2\right]\mathrm{d}\tau\right.\\&-\frac{1}{2}\int_0^1\int_0^1\omega^n(\tau_1)\varepsilon(\tau_1-\tau_2)\omega_n(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\\&+\frac{\alpha}{2}\int_0^1\int_0^1\int_0^1\omega^n(\tau_1)\varepsilon(\tau_1-\tau_2)\mathcal{F}_{nm}(\tau_2)\\&\quad\times\varepsilon(\tau_2-\tau_3)\omega^m(\tau_3)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3\\&-\alpha\theta^n\int_0^1\int_0^1\mathcal{F}_{nm}(\tau_1)\varepsilon(\tau_1-\tau_2)\omega^m(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\\&\left.-\frac{\alpha}{2}\theta^n\theta^m\int_0^1\mathcal{F}_{nm}(\tau_1)\,\mathrm{d}\tau_1\right\}_{\theta=0}.\end{aligned}\qquad(31)$$

The wave function $\Phi(x,\lambda)$ is connected to that of the quadratic Dirac equation $\Psi(x)$ by the relation

$$\Phi(x,\lambda)=\exp\left(-\frac{\mathrm{i}\lambda m^2c^2}{\hbar}\right)\Psi(x).\qquad(32)$$

By analogy with the Schrödinger case, let us calculate the infinitesimal propagation of $\Phi(y,\lambda(\tau))$:

$$\begin{aligned}\Phi(x,\lambda(\tau+d\tau))=&\exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta_{\mathrm{l}}}{\delta\theta^n}\right)\\&\times\int\mathrm{d}y\int\mathrm{d}p\exp\left(\frac{\mathrm{i}}{\hbar}\lambda p^2\,\mathrm{d}\tau\right)\\&\times\int\mathrm{d}\omega(\det(\varepsilon)(\mathrm{d}\tau)^2)^{-1/2}\\&\times\exp\left\{\frac{\mathrm{i}}{\hbar}\left[\frac{-1}{4\lambda}\frac{(x-y)^2}{(\mathrm{d}\tau)^2}-\lambda m^2\omega^2x^2\right]\mathrm{d}\tau\right.\\&-\frac{1}{2}\omega^n(\tau_1)\varepsilon(\tau_1-\tau_2)\omega_n(\tau_2)(\mathrm{d}\tau)^2\\&+\frac{\alpha}{2}\omega^n(\tau_1)\varepsilon(\tau_1-\tau_2)\mathcal{F}_{nm}(\tau_2)\\&\quad\times\varepsilon(\tau_2-\tau_3)\omega^m(\tau_3)(\mathrm{d}\tau)^3\\&-\alpha\theta^n\mathcal{F}_{nm}(\tau_1)\varepsilon(\tau_1-\tau_2)\omega^m(\tau_2)(\mathrm{d}\tau)^2\\&\left.-\frac{\alpha}{2}\theta^n\theta^m\mathcal{F}_{nm}(\tau)\,\mathrm{d}\tau\right\}\Phi(y,\lambda(\tau)),\end{aligned}\qquad(33)$$

with $\tau_1\to\tau_2\to\tau_3$ and $x\to y$ ($\mathrm{d}\tau\to0$). Let us integrate first over the Grassmann variable ω, which is a Gaussian functional integral, the result being

$$\begin{aligned}&\int\mathrm{d}\omega(\det(\varepsilon))^{-1/2}\exp\left\{-\frac{1}{2}\omega^n(\tau_1)\varepsilon(\tau_1-\tau_2)\omega_n(\tau_2)(\mathrm{d}\tau)^2\right.\\&\quad+\frac{\alpha}{2}\omega^n(\tau_1)\varepsilon(\tau_1-\tau_2)\mathcal{F}_{nm}(\tau_2)\varepsilon(\tau_2-\tau_3)\omega^m(\tau_3)(\mathrm{d}\tau)^3\\&\quad\left.-\alpha\theta^n\mathcal{F}_{nm}(\tau_1)\varepsilon(\tau_1-\tau_2)\omega^m(\tau_2)(\mathrm{d}\tau)^2\right\}\\&=[\det(1-\alpha\mathcal{F}\varepsilon\,\mathrm{d}\tau)]^{1/2}\\&\quad\times\exp\left[\frac{1}{2}\alpha^2\theta^n\mathcal{F}_{nk}(1-\alpha\mathcal{F}_{nm}\varepsilon\,\mathrm{d}\tau)^{-1}_{kl}\mathcal{F}_{lm}\varepsilon\theta^m(\mathrm{d}\tau)^2\right].\end{aligned}\qquad(34)$$

As $\det(1-\alpha\mathcal{F}_{nm}\varepsilon\,\mathrm{d}\tau)=\exp(\mathrm{Tr}\ln(1-\alpha\mathcal{F}_{nm}\varepsilon\,\mathrm{d}\tau)$, $\mathrm{Tr}\mathcal{F}=0$ and $\mathrm{Tr}(\varepsilon)=0$ ($\mathcal{F}$ and ε are antisymmetric matrices), and as we can take the development to the first order in $\mathrm{d}\tau$ (the standard Feynman prescription), the result will be equal to 1. This is equivalent to say at first in (33) that $(\mathrm{d}\tau)^3 \ll (\mathrm{d}\tau)^2 \ll \mathrm{d}\tau \to 0$ (the Feynman prescription for the Grassmann dynamical variables).

We now substitute $y = x + \eta$ in (33) taking η small enough in preparation for the series expansion:

$$\begin{aligned}\Phi(x, \lambda(\tau + \mathrm{d}\tau)) &= \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\delta_1}{\delta\theta^n}\right) \\ &\times \int \mathrm{d}\eta \int \mathrm{d}p \exp\left(\frac{\mathrm{i}}{\hbar}\lambda p^2 \,\mathrm{d}\tau\right) \\ &\times \exp\left\{\left[\frac{-\mathrm{i}}{4\lambda\hbar}\frac{(\eta)^2}{\mathrm{d}\tau} - \frac{\mathrm{i}\lambda}{\hbar}(m^2\omega^2 x^2)\,\mathrm{d}\tau\right]\right. \\ &\left. - \frac{\alpha}{2}\theta^n\theta^m\mathcal{F}_{nm}(\tau)\,\mathrm{d}\tau\right\}\Phi(x+\eta, \lambda(\tau)) . \end{aligned} \tag{35}$$

Now we expand it in a power series to the first order in $\mathrm{d}\tau$:

$$\begin{aligned}\Phi(x, \lambda(\tau)) + \mathrm{d}\tau\frac{\partial}{\partial\lambda}\Phi(x, \lambda(\tau)) &= \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\delta_1}{\delta\theta^n}\right) \\ \times \int \mathrm{d}\eta \exp\left(\frac{-\mathrm{i}}{4\lambda\hbar}\frac{(\eta)^2}{\mathrm{d}\tau}\right) &\int \mathrm{d}p \exp\left(\frac{\mathrm{i}}{\hbar}\lambda p^2 \,\mathrm{d}\tau\right) \\ \times \left(1 - \frac{\mathrm{i}\lambda}{\hbar}(m^2\omega^2x^2)\,\mathrm{d}\tau - \frac{\alpha}{2}\theta^n\theta^m\mathcal{F}_{nm}(\tau)\,\mathrm{d}\tau\right) \\ \times \left[\Phi(x, \lambda(\tau)) + \eta\frac{\partial}{\partial x}\Phi(x, \lambda(\tau)) + \frac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}\Phi(x, \lambda(\tau))\right], \end{aligned} \tag{36}$$

and we carry out the Gaussian integration on the variables η and p and the derivation on the θ variables; this gives, after comparison of the left and right hand sides at the same order in $\mathrm{d}\tau$, the Schrödinger type equation verified by $\Phi(x, \lambda(\tau))$:

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial}{\partial\lambda}\Phi(x(t)) &= \left[-\hbar^2\frac{\partial^2}{\partial x^2} + m^2\omega^2\hat{x}^2 - m\omega\hbar\gamma^5\tilde{\gamma}^0\right] \\ &\times \Phi(x, \lambda(\tau)) \\ &= \widetilde{H}\Phi(x, \lambda(\tau)) , \end{aligned} \tag{37}$$

which guarantees (28) and in consequence (3) and (1) are verified.

3 Exact calculation for the Green function

In order to evaluate exactly the Green function expression, we start by integrating over the x_0 variables:

$$\int \mathrm{D}x^0 \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^1 p^0\dot{x}^0\right) = \exp\left\{\mathrm{i}\left(p^0_\mathrm{b}x^0_\mathrm{b} - p^0_\mathrm{a}x^0_\mathrm{a}\right)\right\}\delta(\dot{p}^0) . \tag{38}$$

Next, carrying out the integrations over p^0,

$$\begin{aligned}\int \mathrm{D}p^0 \exp&\left\{\frac{\mathrm{i}}{\hbar}\left(p^0_\mathrm{b}x^0_\mathrm{b} - p^0_\mathrm{a}x^0_\mathrm{a}\right) + \mathrm{i}\lambda\int_0^1 (p^0)^2\,\mathrm{d}\tau\right\}\delta(\dot{p}^0) \\ &= \int \frac{\mathrm{d}p^0}{2\pi\hbar}\exp\left(\frac{\mathrm{i}}{\hbar}p^0\left(x^0_\mathrm{b} - x^0_\mathrm{a}\right) + \frac{\mathrm{i}}{\hbar}\lambda(p^0)^2\right), \end{aligned} \tag{39}$$

we get

$$\begin{aligned}\widetilde{G}^c(x, y) &= \left(\frac{-\mathrm{i}}{\hbar}\right)\int \mathrm{d}\lambda \int \frac{\mathrm{d}p_0}{2\pi\hbar}\int \mathrm{D}x^1 I(\omega) \\ &\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^1 \left[\frac{1}{4\lambda}(\dot{x}^1)^2 + \lambda((p^0)^2 - m^2\right.\right. \\ &\left.\left. - m^2\omega^2(x^1)^2)\right]\mathrm{d}\tau + \frac{\mathrm{i}}{\hbar}p^0\left(x^0_\mathrm{b} - x^0_\mathrm{a}\right)\right\}, \end{aligned} \tag{40}$$

where

$$\begin{aligned}I(\omega) &= \exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta}{\delta\theta^n}\right)\int \mathfrak{D}\omega \\ &\times \exp\left\{-\frac{1}{2}\int_0^1\int_0^1 \omega^n(\tau)\varepsilon(\tau - \acute{\tau})\omega_n(\acute{\tau})\,\mathrm{d}\tau\,\mathrm{d}\acute{\tau}\right. \\ &+ \frac{\alpha}{2}\int_0^1\int_0^1\int_0^1 \omega^n(\tau)\varepsilon(\tau - \acute{\tau})\mathcal{F}_{nm}(\acute{\tau})\varepsilon(\acute{\tau} - \tau'') \\ &\times \omega^m(\acute{\tau})\,\mathrm{d}\tau\,\mathrm{d}\acute{\tau}\,\mathrm{d}\tau'' \\ &- \alpha\theta^n\int_0^1\int_0^1 \mathcal{F}_{nm}(\tau)\varepsilon(\tau - \acute{\tau})\omega^m(\acute{\tau})\,\mathrm{d}\tau\,\mathrm{d}\acute{\tau} \\ &\left.\left. - \frac{\alpha}{2}\theta^n\theta^m\int_0^1 \mathcal{F}_{nm}(\tau)\,\mathrm{d}\tau\right\}\right|_{\theta=0}\end{aligned}$$

is the Polyakov spin factor. In this case, this factor is independent of the x variables and can be carried out explicitly. In order to do that, let us use the following condensed notation:

$$\begin{aligned}I(\omega) &= \exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta}{\delta\theta^n}\right)\int \mathfrak{D}\omega \\ &\times \exp\left\{-\frac{1}{2}\omega^n\varepsilon\omega_n + \frac{\alpha}{2}\omega^n\varepsilon\mathcal{F}_{nm}\varepsilon\omega^m\right. \\ &\left.\left. - \alpha\theta^n\mathcal{F}_{nm}\varepsilon\omega^m - \frac{\alpha}{2}\theta^n\theta^m\mathcal{F}_{nm}\right\}\right|_{\theta=0}, \end{aligned} \tag{41}$$

where

$$\omega^n\varepsilon\omega_n = \int_0^1\int_0^1 \omega^n(\tau)\varepsilon(\tau - \acute{\tau})\omega_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\acute{\tau} . \tag{42}$$

The integration over ω^n has a Gaussian form and gives

$$\begin{aligned}I(\omega) &= \exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta}{\delta\theta^n}\right)\int \mathfrak{D}\omega \\ &\times \exp\left\{-\frac{1}{2}\omega^n\Lambda_{nm}\omega^m + J_n\omega^n\right\}\Big|_{\theta=0} \\ &= \exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta}{\delta\theta^n}\right)\sqrt{\frac{\det\Lambda}{\det\varepsilon}}\exp\left\{-\frac{1}{2}J^n\Lambda^{-1}_{nm}J^m\right\}\Big|_{\theta=0}, \end{aligned} \tag{43}$$

where

$$J_m = -\alpha\theta^n\mathcal{F}_{nm}\varepsilon , \quad \Lambda_{nm} = \eta_{nm}\varepsilon - \alpha\varepsilon\mathcal{F}_{nm}\varepsilon , \tag{44}$$

$$\int \mathfrak{D}\omega^n = \frac{\mathrm{D}\omega^n}{\int \mathrm{D}\omega^n \mathrm{e}^{-\frac{1}{2}\omega^n\varepsilon\omega_n}} , \tag{45}$$

and

$$J^n \Lambda^{-1}_{nm} J^m = -2\alpha^2 \theta^k \mathcal{F}_{kn} G_{nm} \mathcal{F}_{ml} \theta^l , \tag{46}$$

where Λ^{-1}_{nm} is the inverse of Λ_{nm}, and

$$G_{nm} = \frac{1}{2}\varepsilon \Lambda^{-1}_{nm}\varepsilon . \tag{47}$$

On the other hand, we have

$$\det \Lambda = \exp\{\mathrm{Tr} \ln \Lambda\} . \tag{48}$$

So we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\det \Lambda = (\det \Lambda)\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{Tr}\ln \Lambda = (\det \Lambda)\mathrm{Tr}\Lambda^{-1}\frac{\mathrm{d}}{\mathrm{d}\alpha}\Lambda , \tag{49}$$

$$\sqrt{\frac{\det \Lambda}{\det \varepsilon}} = \exp\left\{ -\int_0^\alpha \mathrm{Tr}(G\mathcal{F})\,\mathrm{d}\acute{\alpha}\right\} . \tag{50}$$

Inserting these results in (43), we obtain

$$\begin{aligned} I(\omega) = \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\delta}{\delta\theta^n}\right)\exp\Bigg\{ &-\int_0^\alpha \int_0^1 G^{nm}(\tau,\tau)\mathcal{F}_{nm}\,\mathrm{d}\acute{\alpha} \\ &+ \alpha^2 \int_0^1\int_0^1 \theta^k \mathcal{F}_{kl} G^{lm}(\tau,\acute{\tau})\mathcal{F}_{mn}\theta^n\,\mathrm{d}\tau\,\mathrm{d}\acute{\tau} \\ &- \frac{\alpha}{2}\theta^m\theta^n\mathcal{F}_{mn}\Bigg\}\Bigg|_{\theta=0} . \end{aligned} \tag{51}$$

Now we have to calculate the matrices $\Lambda^{-1}_{\mu\nu}$ and $G_{\mu\nu}$. By definition, $\Lambda_{\mu\nu}(\tau,\acute{\tau})$ and $\Lambda^{-1}_{\mu\nu}(\tau,\acute{\tau})$ verify the property

$$\int_0^1 \Lambda_{\mu\nu}(\tau,s)(\Lambda^{\nu\beta})^{-1}(s,\acute{\tau})\,\mathrm{d}s = \delta^\beta_\mu \delta(\tau-\acute{\tau}) . \tag{52}$$

Using (52), we can conclude (see Appendix A) [19]

$$\begin{aligned} \Lambda^{-1}(\tau,\acute{\tau}) &= \varepsilon^{-1}(\tau,\acute{\tau}) \\ &\quad + (\alpha\mathcal{F})^2 \mathrm{e}^{2\alpha\mathcal{F}(\tau-\acute{\tau})}[\varepsilon(\tau-\acute{\tau}) - \tanh \alpha\mathcal{F}] \\ &\quad + \alpha\mathcal{F}\mathrm{e}^{2\alpha\mathcal{F}(\tau-\acute{\tau})}\delta(\tau-\acute{\tau}) , \end{aligned} \tag{53}$$

$$G(\tau,\acute{\tau}) = \frac{1}{2}\mathrm{e}^{2\alpha\mathcal{F}(\tau-\acute{\tau})}[\varepsilon(\tau-\acute{\tau}) - \tanh \alpha\mathcal{F}] . \tag{54}$$

Substituting (54) into (51), we obtain

$$I(\omega) = \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\delta_\mathrm{l}}{\delta\theta^n}\right)[\cosh(\mathrm{i}m\omega\lambda) - \theta^5\theta^0 \sinh(\mathrm{i}m\omega\lambda)]\big|_{\theta=0} . \tag{55}$$

At this stage, we use the proved formula

$$\exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\delta_\mathrm{l}}{\delta\theta^n}\right) f(\theta)\big|_{\theta=0} = f\left(\frac{\delta_\mathrm{l}}{\delta\xi^n}\right)\exp\left(\mathrm{i}\tilde{\gamma}^n\xi_n\right)\big|_{\xi=0} . \tag{56}$$

Expanding $\exp(\mathrm{i}\tilde{\gamma}^n\xi_n)$ to the second order only (the other terms do not contribute) and then performing differentiation with respect to ξ, we get

$$I(\omega) = [\cosh(\mathrm{i}m\omega\lambda) + \gamma^5\tilde{\gamma}^0 \sinh(\mathrm{i}m\omega\lambda)] . \tag{57}$$

This is the explicit expression of the Polyakov spin factor for the one-dimensional Dirac oscillator. The next step is to carry out the integration over the configuration space,

$$\begin{aligned} K\left(x^1_\mathrm{a}, x^1_\mathrm{b}, \lambda\right) &= \int \mathrm{D}x^1 \\ &\quad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^1 \left(\frac{(\dot{x}^1)^2}{4\lambda} - \lambda m^2\omega^2 (x^1)^2\right)\mathrm{d}\tau\right\} \\ &= \lim_{N\to+\infty}\left(\frac{1}{4\pi\mathrm{i}\hbar\lambda\Delta\tau}\right)^{\frac{N+1}{2}} \int \prod_{k=1}^N \Pi\,\mathrm{d}x^1_k \\ &\quad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{k=1}^{N+1}\left[\frac{\left(x^1_k - x^1_{k+1}\right)^2}{4\lambda\Delta\tau}\right.\right. \\ &\quad\left.\left. - (\Delta\tau)\lambda m^2\omega^2\left(x^1_k\right)^2\right]\right\} . \end{aligned} \tag{58}$$

This expression has the form of a harmonic oscillator propagator [20] with some differences. Taking this into account, we can easily write this propagator as follows:

$$\begin{aligned} K\left(x^1_\mathrm{a}, x^1_\mathrm{b}, \lambda\right) &= \left(\frac{m\omega}{2\pi\mathrm{i}\hbar \sin(2\lambda m\omega)}\right)^{\frac{1}{2}} \\ &\quad\times \exp\left\{\frac{-m\omega}{2\mathrm{i}\hbar}\left[\left(\left(x^1_\mathrm{a}\right)^2 + \left(x^1_\mathrm{b}\right)^2\right)\cot(2\lambda m\omega)\right.\right. \\ &\quad\left.\left. - \frac{2x^1_\mathrm{a}x^1_\mathrm{b}}{\sin(2\lambda m\omega)}\right]\right\} \\ &= \sum_{n\in N_0}\left(\frac{m\omega/\pi\hbar}{(2^n n!)^2}\right)^{\frac{1}{2}} \\ &\quad\times H_n\left(\sqrt{\frac{m\omega}{\hbar}}x^1_\mathrm{a}\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x^1_\mathrm{a}\right) \\ &\quad\times \exp\left\{-\frac{m\omega}{2\hbar}\left(\left(x^1_\mathrm{a}\right)^2 + \left(x^1_\mathrm{b}\right)^2\right)\right\} \\ &\quad\times \mathrm{e}^{-\mathrm{i}2\lambda m\omega\left(n+\frac{1}{2}\right)} . \end{aligned} \tag{59}$$

Turning to the Green function, inserting these latter results, we get

$$\begin{aligned} \widetilde{S}^\mathrm{c}(x,y) &= (\widehat{\tilde{\not{\pi}}} - mc\gamma^5)_\mathrm{out}\left(\frac{-\mathrm{i}}{\hbar}\right)\int \mathrm{d}\lambda \int \frac{\mathrm{d}p_0}{2\pi\hbar} \\ &\quad\times \exp\left(\frac{\mathrm{i}}{\hbar}p^0\left(x^0_\mathrm{b} - x^0_\mathrm{a}\right) + \mathrm{i}\frac{\lambda}{\hbar}\left(p^2_0 - m^2\right)\right) \\ &\quad\times \left[\cosh\left(\frac{\mathrm{i}m\omega e_0}{2}\right) + \gamma^5\tilde{\gamma}^0 \sinh\left(\frac{\mathrm{i}m\omega e_0}{2}\right)\right] \\ &\quad\times K\left(x^1_\mathrm{a}, x^1_\mathrm{b}, \lambda\right) . \end{aligned} \tag{60}$$

Finally, integrating over the proper time λ, multiplying the result by γ^5, choosing for the Dirac matrices the representation

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma^1 = \mathrm{i}\sigma^1 = \mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{61}$$

and acting through the operator $(\widehat{\tilde{\not{\pi}}} - mc\gamma^5)_\mathrm{out}$, the Green function takes the form

$$S^{\mathrm{c}}(x_{\mathrm{a}},x_{\mathrm{b}},T)=-\int\frac{\mathrm{d}p_0}{2\pi\hbar}\sum_{n\in N_0}\left(\frac{m\omega/\pi\hbar}{(2^n n!)^2}\right)^{\frac{1}{2}}$$
$$\times\exp\left\{-\frac{m\omega}{2\hbar}\left(\left(x_{\mathrm{a}}^1\right)^2+\left(x_{\mathrm{b}}^1\right)^2\right)\right\}\frac{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p^0\left(x_{\mathrm{b}}^0-x_{\mathrm{a}}^0\right)}}{p_0^2-p_n^2}$$
$$\times\begin{pmatrix}(p_0+mc)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x_{\mathrm{a}}^1\right)\times H_n\left(\sqrt{\frac{m\omega}{\hbar}}x_{\mathrm{b}}^1\right) & 2n\sqrt{\frac{m\omega}{\hbar}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x_{\mathrm{b}}^1\right)\times H_{n-1}\left(\sqrt{\frac{m\omega}{\hbar}}x_{\mathrm{a}}^1\right)\\ -2n\sqrt{\frac{m\omega}{\hbar}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x_{\mathrm{a}}^1\right)\times H_{n-1}\left(\sqrt{\frac{m\omega}{\hbar}}x_{\mathrm{b}}^1\right) & -2n(p_0-mc)H_{n-1}\left(\sqrt{\frac{m\omega}{\hbar}}x_{\mathrm{a}}^1\right)\times H_{n-1}\left(\sqrt{\frac{m\omega}{\hbar}}x_{\mathrm{b}}^1\right)\end{pmatrix},\tag{62}$$

where

$$c^2p_n^2=(m^2c^4+2n\hbar m\omega c^2)^2,\quad n=0,1,2,\dots\tag{63}$$

4 Wave functions and energy spectrum

To evaluate the wave functions and energy spectrum, let us integrate over the p_0 variable. This can be converted to a complex integral along the special contour C, and then using the residue theorem we get

$$\oint_{\mathrm{C}}\frac{\mathrm{d}p_0}{2\pi\hbar}\frac{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p^0(x_{\mathrm{b}}^0-x_{\mathrm{a}}^0)}}{p_0^2-p_n^2}=-\mathrm{i}\left[\theta(T)\frac{\mathrm{e}^{-\mathrm{i}E_nT}}{2E_n}+\theta(-T)\frac{\mathrm{e}^{\mathrm{i}E_nT}}{2E_n}\right],\tag{64}$$

where

$$E_n=\sqrt{m^2c^4+2n\hbar m\omega c^2}\quad\text{and}\quad cT=x_{\mathrm{b}}^0-x_{\mathrm{a}}^0.\tag{65}$$

In (64), we have two types of propagation, one with positive energy ($+E_n$) propagating to the future and the other with negative energy ($-E_n$) propagating to the past.

From this result, we deduce the energy spectrum and the corresponding wave functions from (62) by writing the following form:

$$S^{\mathrm{c}}(x_{\mathrm{a}},x_{\mathrm{b}},T)=\mathrm{i}\sum_{n\in N_0}\left[\theta(T)\Psi_n^+\left(x_{\mathrm{b}}^1\right)\bar{\Psi}_n^+\left(x_{\mathrm{a}}^1\right)\mathrm{e}^{-\mathrm{i}E_nT}+\theta(-T)\Psi_n^-\left(x_{\mathrm{b}}^1\right)\bar{\Psi}_n^-\left(x_{\mathrm{a}}^1\right)\mathrm{e}^{\mathrm{i}E_nT}\right].\tag{66}$$

This latter is nothing but the spectral decomposition of the propagator from which we identify the wave functions

$$\Psi_n^+(x^1)=\begin{pmatrix}f_n(x^1)\\ g_n(x^1)\end{pmatrix},\quad\Psi_n^-(x^1)=\begin{pmatrix}f_{-n}(x^1)\\ g_{-n}(x^1)\end{pmatrix}.\tag{67}$$

The components of the wave functions Ψ_n^+ and Ψ_n^- are respectively

$$\begin{aligned}f_n(x^1)&=\sqrt{\frac{\sqrt{m\omega/\hbar\pi}}{2^{n+1}n!}\frac{(E_n+mc^2)}{E_n}}\times\exp\left(-\frac{m\omega}{2\hbar}(x^1)^2\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x^1\right),\\ g_n(x^1)&=-\sqrt{\frac{\sqrt{m\omega/\hbar\pi}}{2^n(n-1)!}\frac{(E_n-mc^2)}{E_n}}\times\exp\left(-\frac{m\omega}{2\hbar}(x^1)^2\right)H_{n-1}\left(\sqrt{\frac{m\omega}{\hbar}}x^1\right),\\ f_{-n}(x^1)&=\sqrt{\frac{\sqrt{m\omega/\hbar\pi}}{2^{n+1}n!}\frac{(E_n-mc^2)}{E_n}}\times\exp\left(-\frac{m\omega}{2\hbar}(x^1)^2\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x^1\right)\\ &=\sqrt{\frac{(E_n-mc^2)}{E_n+mc^2}}f_n(x^1),\\ g_{-n}(x^1)&=\sqrt{\frac{\sqrt{m\omega/\hbar\pi}}{2^n(n-1)!}\frac{(E_n+mc^2)}{E_n}}\times\exp\left(-\frac{m\omega}{2\hbar}(x^1)^2\right)H_{n-1}\left(\sqrt{\frac{m\omega}{\hbar}}x^1\right)\\ &=-\sqrt{\frac{(E_n+mc^2)}{(E_n-mc^2)}}g_n(x^1),\end{aligned}\tag{68}$$

where $H_n(z)$ is a Hermite polynomial [21]. We notice that the components of the eigenfunctions obey the relation

$$\int_{-\infty}^{+\infty}\mathrm{d}x[f_n(x)f_m(x)+g_n(x)g_m(x)]=\delta_{nm},\tag{69}$$

with $g_0(x)\equiv 0$. This result agrees exactly with that of [22].

5 Conclusion

In this paper, we succeeded in calculating within the framework of relativistic supersymmetric path integrals the Green function of the one-dimensional Dirac oscillator. The Polyakov spin factor was explicitly evaluated. The remaining integrations over the bosonic variables are of harmonic oscillator type. The exact expression of the causal Green function is obtained, and, using the residue theorem, the spectrum energy and corresponding eigenfunctions expressed in terms of Hermite polynomials are then deduced. Our results coincide with those of the literature. In this one-dimensional case the spin–orbit coupling is not apparent and the calculation was consequently direct and easy. The two-dimensional and three-dimensional cases are of a particular interest because of the spin–orbit coupling, and it would be interesting to extend our calculations to these cases. These problems are under consideration.

Appendix: Inverse matrix $\Lambda_{\mu\nu}^{-1}(\tau,\acute{\tau})$

We are going to calculate the inverse matrix (53). The function $\Lambda_{\mu\nu}$ is defined by

$$\Lambda_{\mu\nu}(\tau,\acute{\tau})=\varepsilon(\tau-\acute{\tau})\eta_{\mu\nu}-\alpha\int_0^1\varepsilon(\tau-\tau_1)\mathcal{F}_{\mu\nu}\varepsilon(\tau_1-\acute{\tau})\,\mathrm{d}\tau_1.\tag{A.1}$$

To get the function $G_{\alpha\beta}$,

$$G_{\alpha\beta}(\tau,\acute{\tau}) = \frac{1}{2}\int_0^1 \Omega_{\alpha\beta}(\tau,\acute{\tau})\varepsilon(s-\acute{\tau})\,\mathrm{d}s\,, \tag{A.2}$$

it is convenient to first define the function $\Omega_{\alpha\beta}$ by

$$\Omega_{\alpha\beta}(\tau,\acute{\tau}) = \int_0^1 \varepsilon(\tau-\lambda)\Lambda^{-1}_{\alpha\beta}(\lambda,\acute{\tau})\,\mathrm{d}\lambda\,. \tag{A.3}$$

Now we have

$$\int_0^1 \Lambda_{\mu\nu}(\tau,s)(\Lambda^{\nu\beta})^{-1}(s,\acute{\tau})\,\mathrm{d}s = \delta^\beta_\mu\delta(\tau-\acute{\tau})\,. \tag{A.4}$$

Substituting (A.1) into (A.4) we obtain

$$\Omega_{\mu\beta}(\tau,\acute{\tau}) - \alpha\int_0^1 \varepsilon(\tau-\tau_1)\mathcal{F}_{\mu\upsilon}\Omega^\nu_\beta(\tau,\acute{\tau})\,\mathrm{d}\tau_1 = \eta_{\mu\beta}\delta(\tau-\acute{\tau})\,. \tag{A.5}$$

This equation is equivalent to the differential equation

$$\frac{\mathrm{d}\Omega_{\mu\beta}(\tau,\acute{\tau})}{\mathrm{d}\tau} - 2\alpha\mathcal{F}_{\mu\upsilon}\Omega^\nu_\beta(\tau,\acute{\tau}) = \eta_{\mu\beta}\frac{\mathrm{d}\delta(\tau-\acute{\tau})}{\mathrm{d}\tau}\,, \tag{A.6}$$

with the initial condition

$$\Omega_{\mu\beta}(0,\acute{\tau}) + \alpha\int_0^1 \mathcal{F}_{\mu\upsilon}\Omega^\nu_\beta(\tau_1,\acute{\tau})\,\mathrm{d}\tau_1 = \eta_{\mu\beta}\delta(\acute{\tau})\,. \tag{A.7}$$

Inserting the general solution

$$\Omega(\tau,\acute{\tau}) = \mathrm{e}^{2\alpha\mathcal{F}\tau}c(\tau,\acute{\tau}) \tag{A.8}$$

into (A.6), we get

$$c(s,\acute{\tau}) = \int_0^s \mathrm{e}^{-2\alpha\mathcal{F}\lambda}\frac{\partial}{\partial\lambda}\delta(\lambda-\acute{\tau})\,\mathrm{d}\lambda + c(\acute{\tau})\,. \tag{A.9}$$

In this way

$$\Omega(\tau,\acute{\tau}) = \mathrm{e}^{2\alpha\mathcal{F}\tau}[c(\acute{\tau})-\delta(\acute{\tau})] + \delta(\tau-\acute{\tau}) + 2\alpha\mathcal{F}\theta(\tau-\acute{\tau})\mathrm{e}^{2\alpha\mathcal{F}(\tau-\acute{\tau})}\,, \tag{A.10}$$

and using the initial condition (A.7), taking into account $\Omega_{\alpha\beta}(0,\acute{\tau}) = c(\acute{\tau})$, we obtain

$$c(\acute{\tau}) - \delta(\acute{\tau}) = \frac{-2\alpha\mathcal{F}\mathrm{e}^{2\alpha\mathcal{F}}}{1+\mathrm{e}^{2\alpha\mathcal{F}}}\mathrm{e}^{-2\alpha\mathcal{F}\acute{\tau}}\,. \tag{A.11}$$

Substituting (A.11) into (A.10) we obtain

$$\Omega(\tau,\acute{\tau}) = \delta(\tau-\acute{\tau}) + \alpha\mathcal{F}\mathrm{e}^{2\alpha\mathcal{F}(\tau-\acute{\tau})}[\varepsilon(\tau-\acute{\tau}) - \tanh\alpha\mathcal{F}]\,. \tag{A.12}$$

Inserting (A.12) into (A.2) results in

$$G(\tau,\acute{\tau}) = \frac{1}{2}\mathrm{e}^{2\alpha\mathcal{F}(\tau-\acute{\tau})}[\varepsilon(\tau-\acute{\tau}) - \tanh\alpha\mathcal{F}]\,. \tag{A.13}$$

According to (A.3), we get

$$\begin{aligned}\Lambda^{-1}(\tau,\acute{\tau}) &= \frac{1}{2}\frac{\partial\Omega(\tau,\acute{\tau})}{\partial\tau}\\ &= \varepsilon^{-1}(\tau,\acute{\tau}) + (\alpha\mathcal{F})^2\mathrm{e}^{2\alpha\mathcal{F}(\tau-\acute{\tau})}\\ &\quad\times[\varepsilon(\tau-\acute{\tau}) - \tanh\alpha\mathcal{F}] + \alpha\mathcal{F}\mathrm{e}^{2\alpha\mathcal{F}(\tau-\acute{\tau})}\delta(\tau-\acute{\tau})\,.\end{aligned} \tag{A.14}$$

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